

THE INCIDENCE CLASS AND THE HIERARCHY OF ORBITS

L. M. FEHÉR AND ZS. PATAKFALVI

ABSTRACT. R. Rimányi defined the incidence class of two singularities η and ξ as $[\eta]|_\xi$, the restriction of the Thom polynomial of η to ξ . He conjectured that (under mild conditions) $[\eta]|_\xi \neq 0 \iff \xi \subset \overline{\eta}$. Generalizing this notion we define the incidence class of two orbits η and ξ of a representation. We give a sufficient condition (positivity) for ξ to have the property that $[\eta]|_\xi \neq 0 \iff \xi \subset \overline{\eta}$ for any other orbit η . We show that for many interesting cases, e.g. the quiver representations of Dynkin type positivity holds for all orbits. In other words in these cases the incidence classes completely determine the hierarchy of the orbits. We also study the case of singularities where positivity doesn't hold for all orbits.

1. INTRODUCTION

Suppose that a complex algebraic representation $\rho : G \rightarrow \mathrm{GL}(V)$ of a complex algebraic Lie group G is given and we want to understand the hierarchy of orbits: which orbit is in the closure of another one. The idea is simple: if η and ξ are two orbits then we calculate $[\eta]$ —the G -equivariant Poincaré dual of $\overline{\eta}$ and restrict it to ξ . Since $[\eta]$ is supported on $\overline{\eta}$, if ξ is disjoint from $\overline{\eta}$, then the incidence class $[\eta]|_\xi$ is zero. If the G -action is rich enough then we have a chance for the opposite implication.

In this paper

- we give a sufficient condition (positivity) for the Incidence Property of an orbit ξ : for any other G -invariant subvariety $X \subset V$ we have that $[X]|_\xi \neq 0 \iff \xi \subset [X]$.
- We show that for many interesting cases, e.g. the quiver representations of Dynkin type positivity holds for all orbits.
- We also study the case of singularities where positivity doesn't hold for all orbits.

Our work was inspired by a conjecture of R. Rimányi in [Rim01], that for singularities of holomorphic maps the incidence classes detect the hierarchy of contact singularity classes.

To study the conjecture we generalized the notion of incidence class to the general group representation setting. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be an algebraic representation of the complex algebraic Lie group G on the vector space V . If $X \subset V$ is a G -invariant subvariety of codimension d then we can assign a G -equivariant cohomology class $[X] \in H_G^{2d}(V) \cong H^{2d}(BG)$. In different areas of mathematics this class has different names e.g. equivariant Poincaré dual, multidegree, Joseph polynomial and—in the case of singularities—Thom polynomial.

Equivariant cohomology shares some properties of ordinary cohomology. For example there is a restriction map: if $Y \subset V$ is another G -invariant subset and $\alpha \in H_G^*(V)$ then $\alpha|_Y \in H_G^*(Y)$. So we can define the incidence class $[X]|_Y$ measuring the “closeness” of X and Y .

Supported by OTKA T046365MAT

Keywords: Thom polynomial, equivariant maps, equivariant Poincaré dual, multidegree, Joseph polynomial, incidence class

AMS Subject classification 32S20.

The crucial observation is that if ξ is an orbit then $[X]|_\xi$ is an equivariant Poincaré dual itself: Let $x \in \xi$ and let us denote by G_x the maximal compact subgroup of the stabilizer group of x . Then $[X]|_\xi \in H_G^*(\xi) \cong H_{G_x}^*(pt)$. Choosing a G_x -invariant normal space N_x to ξ at x we will show that

Proposition 3.1. $[X]|_\xi = [X \cap N_x]_{G_x}$.

This observation reduces the problem to finding a sufficient condition for a representation having the property that all non-empty G -invariant subvariety has a nonzero equivariant Poincaré dual. This condition can be given in terms of the *weights* of the representation. Suppose that $T \subset G$ is a maximal torus. Then V splits into 1-dimensional representations of T with weights $w_i \in \tilde{T} = \text{Hom}(T, U(1))$. If T has rank r then $\tilde{T} \cong \mathbf{Z}^r$.

Definition. The representation $\rho : G \rightarrow \text{GL}(V)$ is *positive* if the convex hull of its weights doesn't contain zero.

Theorem 4.3. *If the representation $\rho : G \rightarrow \text{GL}(V)$ is positive then all non-empty G -invariant subvariety has a nonzero equivariant Poincaré dual.*

From this we immediately get:

Theorem 4.5. *All orbits of the representation $\rho : G \rightarrow \text{GL}(V)$ have the Incidence Property if for all $x \in V$ the normal representation of G_x is positive (normal positivity).*

In section 5. we show a simple condition which implies normal positivity and show that e.g. every quiver representation of Dynkin type satisfy this condition.

After this generalization we faced the problem, familiar to many mathematicians, that our theory doesn't apply to the original conjecture. Most orbits of the representations corresponding to the case of singularities are not positive. Many times it has a trivial reason: the stabilizer is trivial. Even if we restrict ourselves to the orbits having at least a $U(1)$ symmetry, there are plenty of non positive examples. Nonetheless, calculations show that in many cases incidence still detects the hierarchy of orbits. We list some examples, counterexamples and conjectures.

The authors thank M. Kazarian and R. Rimányi for valuable discussions.

2. THE EQUIVARIANT POINCARÉ DUAL AND THE INCIDENCE CLASS

We define equivariant cohomology of a G -space X via the Borel construction and use cohomology with rational coefficients: $H_G^*(X) := H^*(EG \times_G X; \mathbf{Q})$, where $EG \rightarrow BG$ is the universal principal G -bundle over the classifying space BG of G .

We will frequently use the following simple properties of equivariant cohomology:

- (1) $H_G^*(V) \cong H_G^*(pt)$ if V is contractible.
- (2) $H_G^*(X) \cong H_K^*(X)$, where K is a maximal compact subgroup of G . More generally it is true if K is a deformation retract of G .
- (3) We can restrict to an invariant subspace, or to a subgroup. We can also combine them. If the subspace $Y \subset X$ is invariant for a subgroup $S < G$, then we have a restriction map $H_G^*(X) \rightarrow H_S^*(Y)$.

If X is a smooth (always complex in this paper) algebraic variety (typically a complex vector space in this paper) and Y is a G -invariant subvariety then we can assign a class $[Y] \in H_G^*(X)$. For this class not only the names are numerous (some listed in the Introduction) but the definitions, too. These definitions are equivalent for the algebraic setting when the acting group is an algebraic torus. For an account of these results see [Pat06]. Our definition is based on the following fact in ordinary cohomology (see e. g. [Ful97], page 219):

Proposition 2.1 (Definition). *If X is a smooth algebraic variety and Y is an irreducible subvariety of complex codimension d then there is a unique element $[Y] \in H^*(X)$ such that*

- (1) $[Y]$ is supported on Y , i. e. $[Y]$ restricted to $X \setminus Y$ is zero,
- (2) $[Y]|_{X \setminus \text{Sing } Y} = [Y^\circ \subset (X \setminus \text{Sing } Y)]$.

Here $\text{Sing } Y$ denotes the singular subvariety of Y and $Y^\circ = Y \setminus \text{Sing } Y$. The cohomology class $[Y^\circ \subset (X \setminus \text{Sing } Y)]$ is defined by extending the Thom class of a tubular neighbourhood of the proper submanifold $Y^\circ \subset (X \setminus \text{Sing } Y)$ via excision. If Y has many components (of the same codimension) then we take the sum.

This definition can be extended to stratified spaces (stratified by complex submanifolds). For a variety we take the singular stratification: $Y \setminus \text{Sing } Y$, $\text{Sing } Y \setminus \text{Sing}(\text{Sing } Y)$ and so on. Now we define the the equivariant Poincaré dual as a G characteristic class (see [Kaz97]):

Theorem 2.2 (Definition). *Let $\rho : G \rightarrow \text{GL}(V)$ be an algebraic representation and $Y \subset V$ is a G -invariant irreducible subvariety of complex codimension d , then there is a unique element $[Y] \in H_G^*(V)$ such that for all algebraic ρ -bundle $\pi : E \rightarrow M$ over a smooth algebraic variety M with classifying map $k : M \rightarrow BG$*

$$[Y(E) \subset E] = \pi^* k^* [Y],$$

where $Y(E) = P \times_G Y$ for P denoting the principal G -bundle of E .

If the group action is not clear from the context we will use the notation $[Y]_G$ for the G -equivariant Poincaré dual.

Notice that the class $[Y]$ is localized at 0 in the sense that it depends only on the germ of Y at 0. It follows from the fact that the restriction map $H_G^*(V) \rightarrow H_K^*(U)$ is an isomorphism for any K -invariant contractible neighbourhood U of 0. Here we use the maximal compact subgroup K to ensure the existence of small invariant contractible neighbourhoods.

Suppose now that ξ is an orbit of V . Then we have the restriction map $|_\xi : H_G^*(V) \rightarrow H_G^*(\xi)$. Observe that $H_G^*(\xi) \cong H_{G_x}^*(pt)$, where $x \in \xi$ and G_x is the (maximal compact subgroup of the) stabilizer group of x . We can make this more explicit if we restrict to the point x , which is naturally a G_x -space.

It is not difficult to calculate the map $H_G^*(pt) \rightarrow H_{G_x}^*(pt)$. Let $T < G$ be a maximal torus of rank r . Then by the Borel injectivity theorem the restriction map $H_G^*(pt) \rightarrow H_T^*(pt)$ is injective. (This is valid only with rational coefficients. That is the reason we use cohomology with rational coefficients.) So we can identify a class in $H_G^*(pt)$ with a polynomial $p \in \mathbf{Q}[\alpha_1, \dots, \alpha_r] \cong H_T^*(pt)$. We can choose $x \in \xi$ and a maximal torus $T_x < G_x$ such that $T_x < T$. Then restriction of p is a linear substitution in the variables β_1, \dots, β_s , where $H_{T_x}^*(pt) \cong \mathbf{Q}[\beta_1, \dots, \beta_s]$. The substitution is determined by the inclusion $T_x \hookrightarrow T$.

3. THE INCIDENCE CLASS AS AN EQUIVARIANT POINCARÉ DUAL

Since G_x —the maximal compact subgroup of the stabilizer group of x —is a compact group, we can choose a G_x -invariant normal space $N_x < T_x V$ such that $N_x \oplus T_x \xi = T_x V$. We have the exponential map $e_x : N_x \rightarrow V$ which is transversal to ξ . The map e_x is G_x -invariant so it induces a homomorphism $H_G^*(V) \rightarrow H_{G_x}^*(N_x)$.

Proposition 3.1. *Let $Y \subset V$ be a G -invariant subvariety, then $[e_x^{-1}(Y)] = e_x^*[Y]$, or, with some abuse of notation $[Y \cap N_x \subset N_x] = [Y]|_\xi$.*

Proof. It is enough to show that e_x is transversal to Y in a neighbourhood of x . On transversality we mean that e_x is transversal to *every* stratum of Y (see e.g. [Pat06, §2.2]).

The action of G defines a bundle map $\varphi : \mathrm{Lie}(G) \times V \rightarrow TV$. At $x = e_x(0)$ the map e_x is transversal to $\mathrm{Im}(\varphi_x)$, so transversality also holds in an open neighbourhood. But $\mathrm{Im}(\varphi_y)$ is the tangent space of the orbit of y , so in this neighbourhood e_x is transversal to any orbit of G , therefore to any G -invariant submanifold, in particular to the strata of Y . \square

4. THE POSITIVITY CONDITION

In this section we give a condition which implies that every non-empty G -invariant subvariety Y has a non-zero equivariant Poincaré dual.

A simple condition is if the representation $\rho : G \rightarrow \mathrm{GL}(V)$ “contains the scalars”, i.e. the scalars of $\mathrm{GL}(V)$ are in the image of ρ . In this case Y is automatically a cone and its projective degree can be calculated from $[Y]$ by a substitution (see e.g. [FNR07]) so it is never 0.

But the stabilizer group of $x \in V$ typically doesn’t contain the scalars, so we need a more general condition. By the Borel injectivity theorem we can assume that $G = \mathbf{T}$ is a complex torus. This is the point where we use that G is a complex group, so its maximal torus is contained in a complex torus of the same rank. The real torus is too small, it always has orbits with zero equivariant Poincaré dual. Let us denote the set of weights by W_ρ .

Theorem 4.1. *For a representation $\rho : \mathbf{T} \rightarrow \mathrm{GL}(V)$ of the complex torus $\mathbf{T} \cong \mathrm{GL}(1)^r$ the following conditions are equivalent*

- (1) *for all non-empty \mathbf{T} -invariant subvariety $Y \subset V$ the class $[Y] \in H_{\mathbf{T}}^*(V)$ is non-zero.*
- (2) *The convex hull of W_ρ doesn’t contain 0.*

This theorem was first proved in [Pat06, Thm 5.2.1.]. Independently it was noticed in [KS06] that (2) \implies (1) follows from [KM05, Thm D.]. Here we provide a short direct proof. We refer to condition (ii) as *positivity* since it means that there is a linear functional λ on \mathbf{Z}^r such that $\lambda(w) > 0$ for all $w \in W_\rho$. another possible name would be instability as positivity is equivalent to the condition that all points of V are unstable in the GIT sense.

We need one more property of the equivariant Poincaré dual (see [Pat06, prop 4.1]) to prove Theorem 4.1.

Theorem 4.2. *If $Y \subset V$ is a \mathbf{T} -invariant subvariety then the class $[Y] \in H_{\mathbf{T}}^*(V)$ can be expressed as a non-zero polynomial of the weights of ρ with non-negative integer coefficients.*

Theorem 4.2 immediately follows from the fact—well known and widely used by the specialists—that for complex torus actions the notions of equivariant Poincaré dual and *multidegree* coincide. The reason we refer to [Pat06] is that we haven’t found an older proof in the literature. The proof is based on the identification of the multidegree to the equivariant intersection class [EG98], and then the later to the equivariant Poincaré dual. See [KMS06], page 255, for the first equivalence. As for the second one, there is a cycle map from the equivariant Chow group to the equivariant cohomology ring of V ([EG98], page 605). Scrutinizing this map, it turns out that it is in fact the ordinary cycle map of the product of some projective spaces, for which it is easy to prove that it is an isomorphism. A basic fact about multidegree is that the multidegree of a \mathbf{T} -invariant subvariety is equal to the sum of multidegrees of \mathbf{T} -invariant sub(vector)spaces (with possible positive integer multiplicities). Noticing that the equivariant Poincaré dual of a subspace is the

product of some weights of ρ we proved Theorem 4.2. Here we use the identification of a weight $w : \mathbf{T} \rightarrow GL(1)$ with the first Chern class $c_1(L_w)$ of the line bundle $L_w = E \mathbf{T} \times_w \mathbf{C}$.

Proof of Theorem 4.1. We can think of the weights as linear functionals on \mathbf{R}^r —the real part of the Lie algebra of the torus \mathbf{T} . If the convex hull of W_ρ doesn't contain 0 then there is a substitution $\alpha_i \mapsto z_i \in \mathbf{R}$ such that for all $w(\alpha_1, \dots, \alpha_r) \in W_\rho$ we have that $w(z_1, \dots, z_r) > 0$. By Theorem 4.2 the class $[Y]$ is a non-trivial linear combination of monomials of the weights of ρ with non-negative coefficients, so if we apply the same substitution we get a positive number, which implies that $[Y] = p(\alpha_1, \dots, \alpha_r) \neq 0$.

On the other hand, if 0 is in the convex hull, then there is a non-trivial linear combination $\sum n_i w_i = 0$ with $n_i \geq 0$ integers. It implies that $p = \prod x_i^{n_i}$ is a non-zero invariant polynomial (the coordinate x_i corresponds to the one-dimensional eigenspace with weight w_i). Then $Y := \{x \in V : p(x) = 1\}$ is a non-empty invariant subvariety. We have that $0 \notin Y$ so by the fact that $[Y]$ is localized to 0 we get that $[Y] = 0$. \square

Theorem 4.1 immediately implies

Theorem 4.3. *If the representation $\rho : G \rightarrow GL(V)$ is positive then all non-empty G -invariant subvariety has a nonzero equivariant Poincaré dual.*

Let us recall now the definition of Incidence Property from the Introduction.

Definition 4.4. *Given a representation $\rho : G \rightarrow GL(V)$ we say that an orbit ξ has the Incidence Property if for any other G -invariant subvariety $X \subset V$ we have that $[X]|_\xi \neq 0 \iff \xi \subset [X]$.*

Proposition 3.1 and Theorem 4.3 now implies that

Theorem 4.5. *All orbits of the representation $\rho : G \rightarrow GL(V)$ have the Incidence Property if for all $x \in V$ the normal representation of G_x is positive (normal positivity).*

5. EXAMPLES OF REPRESENTATIONS WITH THE INCIDENCE PROPERTY

First we give a stronger condition than positivity, which is easier to check. The examples we have in mind are sub-representations of the left-right action of a subgroup of $GL(W) \times GL(W)$ on the vector space $\text{Hom}(W, W)$. In these cases the weights are of the form $t - s$ where t is a “target” weight and s is a “source” weight.

Proposition 5.1. *Suppose that $\rho : G \rightarrow GL(V)$ is a complex representation, T is a maximal torus of G and there are linearly independent elements $e_1, \dots, e_s \in \check{T}$ such that all weights of ρ are of the form $e_i - e_j$ for some $i < j$. Then ρ is positive, i.e. 0 is not in the convex hull of the weights of ρ .*

Proof. It is enough to check that 0 is not in the convex hull of $\{e_i - e_j : i < j\}$. Assume that $\sum n_{ij}(e_i - e_j) = 0$ for some non-negative integers n_{ij} . But $\sum n_{ij}(e_i - e_j) = \sum m_k e_k$ for some integers m_k with $m_1 = \sum n_{1j}$. It implies that $n_{1j} = 0$ for all j . An induction on s finishes the proof. \square

5.1. Schubert-varieties of flag manifolds. Now we show the Incidence Property to the representation discussed in [FR03] and [KM05].

Let the group $B^+ \times B^-$ act on $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \cong \{(m_{ij})_{i,j=1}^n\}$, where B^+ and B^- are the groups of $n \times n$ upper and lower triangular matrices, respectively, and $(R, L) \cdot M = LMR^{-1}$ for $R \in B^+$, $L \in B^-$ and $M \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.

Every orbit contains a unique 0-1 matrix A , every column and row of which contains at most one 1. (Note that the full rank matrices correspond to the Schubert varieties of the n -dimensional flag-manifold.) We can expand such a matrix uniquely to the right and down to a $2n \times 2n$ permutation matrix. This expansion is obtained by adding 1's to rows not containing 1's starting from the top, putting the 1 in the left-most position, where it is possible, outside of the upper-left $n \times n$ matrix. This way we can associate a permutation $\pi \in S_{2n}$ to an orbit, by setting $\pi(i) = j$, if there is a 1 in the (i, j) position of the permutation matrix obtained.

According to [FR03, §4.] the maximal torus of the stabilizer group of the matrix A with expanded permutation π is

$$T_\pi = \{(\text{diag}(\alpha_1, \dots, \alpha_n), \text{diag}(\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)})) \mid \alpha_i \in U(1)\}$$

and an invariant normal space to the orbit of A is

$$N_A = \{(m_{ij})_{i,j=1}^n \mid m_{ij} = 0 \text{ if } \pi(i) \leq j \text{ or } \pi^{-1}(j) \leq i\}$$

of $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ invariant under G_π action.

Using the basis e_1, \dots, e_n for $\tilde{T}_\pi \cong U(1)^n$ the weights of N_A are of the form $e_{\pi(i)} - e_j$ where $\pi(i) > j$, so by Proposition 5.1. (for the reverse ordering) this orbit has the Incidence Property.

Equivalent statements were proved in [LS92], [Kum96], [Gol01] and [BR04] as a characterization of the *Bruhat order*. In fact a formula in [BR04] gave us the idea to study the relation of incidence with multidegree.

5.2. Quiver representations. In this section we show that for representations corresponding to quivers of Dynkin type all orbits have the Incidence Property. The equivariant Poincaré duals for these cases were first calculated in [FR02a]. It also contains a more detailed introduction to the geometry of these representations. Recently another algorithm was given in [KS06].

Consider an oriented graph Q and denote by Q_0 the set of its vertices and by Q_1 the set of its edges. If e is an edge of Q , then let e' and e'' denote the head and the tail of e , respectively.

If a function $d : Q_0 \rightarrow \mathbb{N}$ (dimension vector) is given we have the group $G = \bigoplus_{i \in Q_0} \text{GL}(d(i))$

acting on $V = \bigoplus_{e \in Q_1} \text{Hom}(\mathbb{C}^{d(e')}, \mathbb{C}^{d(e'')})$, by the formula

$$\left(\bigoplus_{i \in Q_0} A_i \right) \cdot \left(\bigoplus_{e \in Q_1} \varphi_e \right) = \bigoplus_{e \in Q_1} (A_{e''} \varphi_e A_{e'}^{-1})$$

The orbits of this representation correspond to the representations (modules) of the path algebra $\mathbb{C}Q$, with dimension vector d .

For graphs of Dynkin type, the set $R(Q)$ of indecomposable modules of $\mathbb{C}Q$ is finite. Any $\mathbb{C}Q$ -module can be decomposed into a form of $\sum_{r \in R(Q)} \mu_r r$ for some integer numbers μ_r by the Krull-Schmidt theorem and the numbers μ_r are well defined. The maximal compact stabilizer

subgroup of M is $G_M = \bigoplus_{r \in R(Q)} U(\mu_r)$. A normal space is $N_M = \bigoplus_{r,s \in R(Q)} \text{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})^{m_{rs}}$, where $m_{rs} = \dim \text{Ext}_{\mathbb{C}Q}(r, s)$, and G_M acts on N_M with the rule

$$\left(\bigoplus_{r \in R(Q)} A_r \right) \cdot \left(\bigoplus_{r,s \in R(Q)} \bigoplus_{i=1}^{m_{rs}} \varphi_{rs}^i \right) = \left(\bigoplus_{r,s \in R(Q)} \bigoplus_{i=1}^{m_{rs}} A_s \varphi_{rs}^i A_r^{-1} \right)$$

where $\varphi_{rs}^i \in \text{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})$ for $i = 1, \dots, m_{rs}$.

The maximal torus T_M of the stabilizer group of M is isomorphic to $U(1)^{\sum \mu_r}$. Let us denote the standard basis of the weight lattice \tilde{T}_M by $\{e_{r,j} : r \in R(Q), j \leq \mu_r\}$. Then the weights of the representation of G_M on $\text{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})$ are $e_{s,i} - e_{r,j}$ for any $1 \leq i \leq \mu_s$ and $1 \leq j \leq \mu_r$. To apply Proposition 5.1 it is enough to show that there is an ordering \succ on $R(Q)$, such that if $m_{rs} \neq 0$, then $r \succ s$.

We recall the notion of the *Auslander-Reiten translate* τ which is partial self mapping map of $R(Q)$ with the following properties.

- (1) For every $r \in R(Q)$, there is a unique $n_r \in \mathbb{N}$, such that $\tau^{n_r} r$ is projective. (In other words: in our case all indecomposables are pre-projective.)
- (2) $\text{Ext}_{\mathbb{C}Q}(r, s) = \text{Ext}_{\mathbb{C}Q}(\tau r, \tau s)$ for every $r, s \in R(Q)$, where both τr and τs are defined.

(1) implies that there is an ordering \succ on $R(Q)$, for which if $n_r > n_s$, then $r \succ s$. Suppose that $r \not\succ s$. Then $n_r \leq n_s$, and consequently $m_{rs} = \dim \text{Ext}_{\mathbb{C}Q}(l_r, l_s) = \dim \text{Ext}_{\mathbb{C}Q}(\tau^{n_r} l_r, \tau^{n_r} l_s) = 0$, since $\tau^{n_r} l_r$ is projective. This shows that indeed if $m_{rs} \neq 0$, then $r \succ s$, and we proved

Theorem 5.2. *For a representation corresponding to a Dynkin type quiver all orbits satisfy the Incidence Property.*

6. SINGULARITIES

In this section we study incidences of singularities of holomorphic map germs. On singularity we mean a contact orbit of map germs. The equivariant Poincaré dual in this context is called *Thom polynomial*. We use [Rim01] and [FR04] as a general reference.

The space $\mathcal{E}(n, p)$ of holomorphic map germs from \mathbb{C}^n to \mathbb{C}^p and the contact group $\mathcal{K}(n, p)$ acting on it are infinite dimensional. To define G -equivariant cohomology for infinite dimensional groups some extra care is needed as it is explained in [Rim01]. Our main interest lies in studying *finite codimensional* singularities where a finite dimensional reduction is available.

Only finite codimensional singularities have Thom polynomials so it is a natural choice. It is possible to restrict to infinite codimensional singularities (for example see [FR02b]), but to simplify the situation we stick with the finite codimensional case.

Every finite codimensional singularity η is *finitely determined*, i.e. there is a $k \in \mathbb{N}$ such that for any two germs $f \in \eta$ and $g \in \mathcal{E}(n, p)$ we have that $g \in \eta$ if and only if their k^{th} Taylor polynomials (or k -jets) $j^k(f)$ and $j^k(g)$ are contact equivalent. So we can reduce to the finite dimensional space of k -jets: $J^k(n, p) \cong \bigoplus_{i=1}^k \text{Hom}(\text{Sym}^i \mathbb{C}^n, \mathbb{C}^p)$. We have the truncating map $t_k : \mathcal{E}(n, p) \rightarrow J^k(n, p)$. Similarly we can define $\mathcal{K}^k(n, p)$, the group of k -jets of the contact group:

$$\mathcal{K}^k(n, p) := \{(\varphi, \psi) \in J^k(n, n)^\times \times J^k(n + p, n + p)^\times : \psi|_{\mathbb{C}^n \times 0} = \text{Id}_{\mathbb{C}^n}, \pi_{\mathbb{C}^n} \psi = \psi \pi_{\mathbb{C}^n}\}.$$

This group acts on $J^k(n, p)$ and for the truncating homomorphism $\pi_k : \mathcal{K}(n, p) \rightarrow \mathcal{K}^k(n, p)$ the map t_k is π_k -equivariant. The homomorphism π_k induces isomorphism on equivariant cohomology

and for any two k -determined singularities η and ξ :

$$[\eta]|_\xi = [t_k \eta]|_{t_k \xi}.$$

We don't prove this statement—as we avoided defining the left hand side—but use as a motivation to study the representation of the algebraic group $\mathcal{K}^k(n, p)$ on the jet space $J^k(n, p)$.

We cannot expect that all orbits of this representation have the Incidence Property. There are many singularities ξ with small symmetry: the maximal torus of their stabilizer group is trivial. It implies that the restriction map is automatically trivial. To have a chance for the Incidence Property to be satisfied we have to require the existence of at least a $U(1)$ symmetry. Since the diagonal torus $U(1)^n \times U(1)^p$ is maximal in $\mathcal{K}^k(n, p)$ and all maximal tori are conjugate, this requirement implies that a representative $f = (f_1, \dots, f_p)$ of the orbit ξ with diagonal symmetry can be chosen. Then f is a *weighted homogeneous* polynomial, i.e. there are weights $a_1, \dots, a_n, b_1, \dots, b_p \in \mathbf{Z}$ such that

$$f_j(t^{a_1} x_1, \dots, t^{a_n} x_n) = t^{b_j} f_j(x_1, \dots, x_n)$$

for $j = 1, \dots, p$. These polynomials are also called quasi-homogeneous.

The Incidence Property can fail even for weighted homogeneous polynomials, which can be detected by restricting them to themselves. According to the definition. Namely if ξ has the Incidence Property then $e(\xi) = [\xi]|_\xi \neq 0$.

Example 6.1. For $\xi = (x^2 + 3yz, y^2 + 3xz, z^2 + 3xy)$ the normal Euler class $e(\xi) = 0$.

Remark 6.2. This Euler class can be directly calculated using unfolding—see the second part of this section—but there is a deeper reason for the vanishing of $e(\xi)$. This orbit is a member of a one-parameter family of orbits: $f_\lambda = (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy)$. (This is a famous example ([Wal77]), the smallest codimensional occurrence of a family in the equidimensional case.) The tangent direction of the family in the normal space N_ξ has 0 weight, which implies that $e(\xi) = 0$. In fact we don't know any example when $e(\xi) = 0$ but ξ is *not in a family*.

The following conjecture is a slightly modified version of Rimányi's from [Rim01]:

Conjecture 6.3. *The singularity $\xi \subset J^k(n, p)$ has the Incidence Property iff $e(\xi) \neq 0$.*

Remark 6.4. Defining the Incidence Property (Definition 4.4) in such a way that we require the condition $e(\xi) = [\xi]|_\xi \neq 0$ to be satisfied seems formal, we know that ξ is a subset of $\bar{\xi}$ anyway. However the vanishing of $e(\xi)$ sometimes can be related to the vanishing of a proper incidence $[\eta]|_\xi$ where $\eta \neq \xi$ but $\xi \subset \bar{\eta}$. Consider the following abstract example:

Example 6.5. Let $GL(1)$ act on \mathbf{C}^2 by $z(x, y) = (x, zy)$. The orbits of this representation are $f_\lambda = \{(\lambda, 0)\}$ and $g_\lambda = \{(\lambda, y) : y \neq 0\}$. The fixed point f_λ has a $U(1)$ symmetry and $f_\lambda \subset \overline{g_\lambda}$ but the incidence $[g_\lambda]|_{f_\lambda}$ is zero. We have a geometric reason for this: $[g_\mu]|_{f_\lambda} = 0$ if $\mu \neq \lambda$ since $f_\lambda \not\subset \overline{g_\mu}$. But $[g_\mu]$ depends continuously on μ so it is constant. We can also calculate directly: $\overline{g_\lambda}$ is a line so $[g_\lambda]|_{f_\lambda}$ is the Euler class of the complementary invariant line $\{(\lambda, 0) : \lambda \in \mathbf{C}\}$. But this is exactly the set of fixed points so the Euler class is zero.

We believe that such situation is not uncommon for singularities, i.e. there are contact orbits f_λ and g_λ depending on the parameter λ continuously such that $\text{codim}(f_\lambda) = \text{codim}(g_\lambda) + 1$, $f_\lambda \subset \overline{g_\lambda}$ and f_λ has a $U(1)$ symmetry. The same reasoning shows that the incidence $[g_\lambda]|_{f_\lambda}$ is zero in this case. Unfortunately we were unable to verify the existence of such an example.

Theorem 4.3 implies that

Proposition 6.6. *If a singularity $\xi \subset J^k(n, p)$ is positive i.e the normal action of the stabilizer group is positive, then ξ has the Incidence Property.*

Complicated singularities are usually not positive:

Example 6.7. The singularities (x^a, y^b) for $2 \leq a \leq b$ are positive exactly for (x^2, y^2) , (x^2, y^3) and (x^2, y^4) .

On the other hand we know many positive cases:

Example 6.8. All Σ^1 and $\Sigma^{2,0}$ singularities (type A_n , $I_{a,b}$, $III_{a,b}$, IV_a , V_a , for the notation see [Mat71]) are positive.

Remark 6.9. In fact all the incidences of the singularities in Example 6.8 can be explicitly calculated since their adjacencies are well understood (see [Rim99]). For example in the equidimensional case for $2 \leq a \leq b$, $(a, b) \neq (2, 2)$, $2 \leq c \leq d$ and $c + d < a + b$ we have [Rim]:

$$[I_{c,d}]|_{I_{a,b}} = (a-1)!(b-1)! \left(\frac{a^d b^c}{(a-c)!(b-d)!} + \frac{a^c b^d}{(a-d)!(b-c)!} \right) \cdot \frac{1}{\delta_{cd} + 1} \cdot g,$$

where $k! = \infty$ for k negative and $\delta_{cd} = 1$ if $c = d$ and 0 otherwise. The maximal torus of the stabilizer group of $I_{a,b}$ for $(a, b) \neq (2, 2)$ is $U(1)$ and g denotes the generator of $\mathbf{Z} \cong H_{U(1)}^D(pt; \mathbf{Z})$ where D is the degree of the cohomology class $[I_{c,d}]$.

We have one result which goes beyond the positivity condition. To state it, we recall the definition of Σ^i classes:

Definition 6.10.

$$\Sigma^i := \Sigma^i(n, p) = \{f \in J^k(n, p) : \dim \ker d_0 f = i\}.$$

We suppressed the dependence on k in the notation as the Σ^i -class of f depends only on the linear term $j^1 f$.

Theorem 6.11. *Suppose that the contact orbit $\xi \subset \Sigma^0 \cup \Sigma^1 \cup \Sigma^2$ has at least a $U(1)$ symmetry (i.e. there is a weighted homogeneous polynomial in the orbit). Then the incidence $[\Sigma^2]|_\xi$ vanishes if and only if $\xi \notin \Sigma^2$.*

Notice that the classification of Σ^2 singularities is not known, so case by case checking is not available here. To prepare the proof we make some remarks.

Remarks 6.12.

- (i) If the contact orbit ξ is in $\Sigma^0 \cup \Sigma^1 \cup \Sigma^2$ then ξ has a representative f such that f depends only on the first two variables so we will assume that $n = 2$. If $f \in J^k(2, p)$ then $f \in \Sigma^2$ if and only if f has no linear terms.
- (ii) Σ^2 always contains a contact orbit which is open in Σ^2 . If $k = 1$ then Σ^2 is an orbit. If $k \geq 2$ and $p = 2$ then $I_{2,2}$ —the orbit of (x^2, y^2) —is open in Σ^2 . If $p > 2$ then $III_{2,2}$ —the orbit of $(x^2, xy, y^2, 0, \dots, 0)$ —is open in Σ^2 . So for example for the latter case $[III_{2,2}] = [\Sigma^2]$ (see [Mat71]).

(iii) By the Thom-Porteous-Giambelli formula [Por71] we have

$$[\Sigma^2(2, p)] = \prod (\beta_i - \alpha_1)(\beta_i - \alpha_2)$$

in terms of Chern roots. In other words $\alpha_1, \alpha_2, \beta_1, \dots, \beta_p$ denote the generators of $H_T^2(pt)$ where $T = U(1)^2 \times U(1)^p$ is the maximal torus acting on $J^1(2, p)$. (It can also be seen directly since $J^1(2, p) \cap \Sigma^2(2, p) = 0$.)

(iv) Consequently for the weighted homogeneous polynomial $f \in J^k(2, p)$ with one $U(1)$ symmetry, i.e. with stabilizer subgroup of rank one, and with weights a_1, a_2 and b_1, \dots, b_p we have

$$(1) \quad [\Sigma^2]_f = \prod (b_i - a_1)(b_i - a_2)g^{2p}$$

where g is the generator of the cohomology ring $H_{U(1)}^2(pt; \mathbf{Z}) \cong \mathbf{Z}[g]$ of the symmetry group. In other words the incidence is zero if and only if there is a target weight equal to a source weight.

If f has symmetry group $U(1) \times U(1)$, then we can still restrict $[\Sigma^2]_f$ further to the subtorus U of $U(1) \times U(1)$ corresponding to the weighting of f . Then we get formula (1) for $([\Sigma^2]_f)|_U$. Hence if no target weight is equal to a source weight we still get that the incidence is not zero.

(v) Notice that (1) holds only if f has no additional symmetry. For example if f is contact equivalent to a monomial germ then the rank of the symmetry group is at least 2. If $f \in \Sigma^2(2, p)$ is monomial then the incidence $[\Sigma^2]_f$ is not zero since we can restrict to the $U(1)$ symmetry corresponding to the source weights $a_1 = a_2 = 1$ and see that $b_i \geq 2$ for all $i \leq p$ since $f \in \Sigma^2(2, p)$ implies that there are no linear terms.

These remarks imply that to prove Theorem 6.11 it is enough to prove:

Proposition 6.13. *Let $f \in J^k(2, p)$ be a weighted homogeneous polynomial with weights a_1, a_2 and b_1, \dots, b_p . If $a_1 = b_1$ then f is contact equivalent to a monomial germ.*

Proof. We distinguish three cases and in all cases we will use the fact that if the ideals (f_1, \dots, f_p) and (f'_1, \dots, f'_p) agree then f is contact equivalent to f' . In the first two cases we don't need the $a_1 = b_1$ assumption.

Case 1: If $a_1 = 0$ or $a_2 = 0$. If, say, $a_2 = 0$ then $f_i = x^{b_i/a_1}g_i(y)$ for some polynomials $g_i(y)$. If the lowest power of y in $g_i(y)$ is k_i then $g_i(y) = y^{k_i}(h_i + y\bar{g}_i(y))$ for some polynomials $\bar{g}_i(y)$ and non-zero numbers h_i . The functions $h_i + y\bar{g}_i(y)$ are units in the algebra $\mathbf{C}[y]/(y^{k+1})$ which implies that the ideal (f_1, \dots, f_p) is equal to $(x^{b_1/a_1}y^{k_1}, \dots, x^{b_p/a_1}y^{k_p})$ i.e. f is contact equivalent to a monomial germ.

Case 2: If $a_1 > 0$ and $a_2 < 0$. Suppose that $\deg x^u y^v = \deg x^{u'} y^{v'}$ and $u \geq u'$. Then $v \geq v'$. It implies that $f_i(x, y) = x^{u_i} y^{v_i} (h_i + g_i(x, y))$ for some u_i, v_i non-negative integers and $h_i \neq 0$ constant. Therefore $(f_1, \dots, f_p) = (x^{u_1} y^{v_1}, \dots, x^{u_p} y^{v_p})$.

Case 3: If $a_1 > 0$ and $a_2 > 0$. Then $a_1 = b_1$ implies that $f_1 = y^k$ for $a_1 = ka_2$. Without loss of generality we can assume that $a_1 = k$ and $a_2 = 1$. From here we don't use the $a_1 = b_1$ assumption. There is a unique decomposition $b_i = u_i k + d_i$ for $0 \leq d_i < k$. Then $(f_i, y^k) = (x^{u_i} y^{d_i}, y^k)$ for $2 \leq i \leq p$, therefore $(f_1, \dots, f_p) = (y^k, x^{u_2} y^{d_2}, \dots, x^{u_p} y^{d_p})$. \square

6.1. Calculations of the Examples 6.7 and 6.8. Let us recall now that the normal space of the contact orbit of the singularity $\xi = (f_1, \dots, f_p)$ is isomorphic to $\mathbf{C}^n \oplus U_\xi$ where $U_\xi = J^k(n, p)/(f_i e_j, \partial f/x_l : i, j \leq p, l \leq n)$ is the *unfolding space* of ξ . Here e_j denotes the constant map $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the j^{th} coordinate. (see e.g. in [AGZV88] or [FR04]).

(x^a, y^b) : The unfolding space is spanned by the monomials $\{(x^i y^j, 0) : i < a - 1, j < b\}$ and $\{(0, x^i y^j) : i < a, j < b - 1\}$ (except for $i = 0, j = 0$ —constants are not in $J^k(n, p)$). The maximal torus of the symmetry is a $U(1)^2 = \{(\alpha, \beta) : \alpha, \beta \in U(1)\}$ acting via $((\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}), (\begin{smallmatrix} \alpha^a & 0 \\ 0 & \beta^b \end{smallmatrix}))$ on $\text{Hom}(\mathbf{C}^2, \mathbf{C}^2)$ therefore the weights of the normal space are

$$\{(i, j) : i = 2, \dots, a; j = -b + 1, \dots, 0\} \setminus \{(a, 0)\}$$

and

$$\{(i, j) : i = -a + 1, \dots, 0; j = 2, \dots, b\} \setminus \{(0, b)\}.$$

By sketching the distribution of the weights on the coordinate plane we can see that $(0, 0)$ is in the convex hull unless $a = 2$ and $b = 2, 3$ or 4 .

To demonstrate that Σ^1 and $\Sigma^{2,0}$ singularities are positive we pick the series $III_{a,b}$. the other calculations are similar but simpler since the symmetry is $U(1)$.

$III_{a,b} = (x^a, xy, y^b)$: The unfolding space is spanned by

$$\{(x^j, 0, 0) : j = 1, \dots, a - 2\} \cup \{(y^j, 0, 0) : j = 1, \dots, b - 1\} \cup$$

$$\{(0, 0, x^j) : j = 1, \dots, a - 1\} \cup \{(0, 0, y^j) : j = 1, \dots, b - 2\} \cup \{(ax^{a-1}, y, 0), (0, x, by^{b-1})\}.$$

The symmetry is $U(1)^2$ and the weights are

$$\{(1, 0), (2, 0), \dots, (a-1, 0); (a, -1), (a, -2), \dots, (a, 1-b); (0, 1), \dots, (0, b-1); (-1, b), \dots, (1-a, b)\}.$$

The linear functional $bx + ay$ is positive on all these weights proving the positivity of $III_{a,b}$.

REFERENCES

- [AGZV88] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1988.
- [BR04] Anders S. Buch and Richárd Rimányi. Specializations of Grothendieck polynomials. *C. R. Math. Acad. Sci. Paris*, 339(1):1–4, 2004.
- [EG98] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.
- [FNR07] László M. Fehér, András Némethi, and Richárd Rimányi. The degree of the discriminant of irreducible representations. to appear in *J. Algebraic Geometry*, 2007.
- [FR02a] László Fehér and Richárd Rimányi. Classes of degeneracy loci for quivers: the Thom polynomial point of view. *Duke Math. J.*, 114(2):193–213, 2002.
- [FR02b] László M. Fehér and Richárd Rimányi. Thom polynomials with integer coefficients. *Illinois J. Math.*, 46(4):1145–1158, 2002.
- [FR03] László M. Fehér and Richárd Rimányi. Schur and Schubert polynomials as Thom polynomials—cohomology of moduli spaces. *Cent. Eur. J. Math.*, 1(4):418–434 (electronic), 2003.
- [FR04] László M. Fehér and Richárd Rimányi. Calculation of Thom polynomials and other cohomological obstructions for group actions. In *Real and complex singularities*, volume 354 of *Contemp. Math.*, pages 69–93. Amer. Math. Soc., Providence, RI, 2004.
- [Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
- [Gol01] R. F. Goldin. The cohomology ring of weight varieties and polygon spaces. *Adv. in Math.*, 160:175–204, 2001.

- [Kaz97] M. É. Kazarian. Characteristic classes of singularity theory. In *The Arnold-Gelfand mathematical seminars*, pages 325–340. Birkhäuser Boston, Boston, MA, 1997.
- [KM05] Allen Knutson and Ezra Miller. Gröbner geometry of Schubert polynomials. *Annals of Math.*, 2(3):1245–1318, 2005. math.AG/0110058.
- [KMS06] Allen Knutson, Ezra Miller, and Mark Shimozono. Four positive formulae for type A quiver polynomials. *Invent. math.*, 166:229–325, 2006.
- [KS06] Allen Knutson and Mark Shimozono. Kempf collapsing and quiver loci. math.AG/0608327, 2006.
- [Kum96] S. Kumar. The nil hecke ring and singularity of Schubert varieties. *Invent. Math.*, 123(3):471–506, 1996.
- [LS92] A. Lascoux and M.-P. Schützenberger. Décompositions dans l’algèbre des differences divisées. *Discrete Math.*, 99(1–3):165–179, 1992.
- [Mat71] J. Mather. Stability of C^∞ mappings. VI. the nice dimensions. In *Liverpool Singularities — Symposium I*, number 192 in SLNM, pages 207–253, 1971.
- [Pat06] Zsolt Patakfalvi. Orbit structures and incidence. Master’s thesis, Eotvos University, Budapest, 2006. <http://www.cs.elte.hu/math/diploma/math/kopasz.pdf>.
- [Por71] I. Porteous. Simple singularities of maps. In *Liverpool Singularities—Symposium I*, number 192 in SLNM, pages 286–307, 1971.
- [Rim] Richárd Rimányi. private communication.
- [Rim99] Richárd Rimányi. *Generalized Pontrjagin-Thom construction for singular maps*. PhD thesis, Eotvos University, Budapest, 1999. http://www.cs.elte.hu/phd_th/rimanyi.ps.gz.
- [Rim01] Richárd Rimányi. Thom polynomials, symmetries and incidences of singularities. *Invent. Math.*, 143(3):499–521, 2001.
- [Wal77] C. T. C. Wall. Nets of conics. *Math. Proc. Cambridge Philos. Soc.*, 81(3):351–364, 1977.

DEPARTMENT OF ANALYSIS, EOTVOS UNIVERSITY BUDAPEST, HUNGARY, RÉNYI INSTITUTE BUDAPEST, HUNGARY

E-mail address: lfeher@renyi.hu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA-98195, USA

E-mail address: pzs@math.washington.edu